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CONSTRUCTION OF BASIS FUNCTIONS AND THEIR APPLICATION TO BOUNDARY-VALUE PROBLEMS OF MECHANICS OF CONTINUOUS MEDIA

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A unified method for constructing basis (eigen) functions is proposed to solve problems of mechanics of continuous media, problems of cubature and quadrature, and problems of approximation of hypersurfaces. Numerical-analytical methods are described, which allow obtaining approximate solutions of internal and external boundary-value problems of mechanics of continuous media of a certain class (both linear and nonlinear). The method is based on decomposition of the sought solutions of the considered partial differential equations into series in basis functions. An algorithm is presented for linearization of partial differential equations and reduction of nonlinear boundary-value problems, which are reduced to systems of linear algebraic equations with respect to unknown coefficients without using traditional methods of linearization.

Key words: *basis functions, boundary-value problem, linearization, invariant solutions, continuous medium.*

In the present work, we develop methods for solving linear and nonlinear boundary-value problems of mechanics of continuous media on the basis of global or local approximation of the sought solutions of equations and boundary conditions by functions found by expansion of the solutions into series in terms of basis functions.

1. Construction of Basis Functions. The basis functions presented below are constructed on invariant solutions of partial differential equations that admit a group of extensions (compressions) in terms of dependent and independent variables and a group of translations in terms of independent variables. The fundamental works related to construction of invariant solutions and basis functions are [1–4]. We consider the two-dimensional Laplace equation

$$\Delta U \equiv \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0. \tag{1.1}$$

The invariant solution of Eq. (1.1) is sought in the form

$$U = x^\alpha J(\eta), \qquad \eta = y/x, \tag{1.2}$$

where α is an arbitrary real number. Substituting (1.2) into (1.1), we obtain

$$x^{\alpha-2}[(\eta^2+1)J'' - 2\eta(\alpha-1)J' + \alpha(\alpha-1)J] = 0, \tag{1.3}$$

where J' and J'' are the first and second derivatives with respect to η and $x^{\alpha-2} \neq 0$.

The solution of the differential equation (1.3) is sought in the form of the series

$$J = \sum_{k=0}^{\infty} c_k \eta^k. \tag{1.4}$$

Substituting (1.4) into (1.3) and equating the coefficients at identical powers of η, we find the recurrent formula

$$c_{k+2} = -\frac{(\alpha-k)(\alpha-k-1)}{(k+2)(k+1)}\, c_k,$$

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which allows us to express all even coefficients of series (1.4) via c_0 and all odd coefficients via c_1. We choose c_0 and c_1 as coefficients forming the initial basis. Then, we obtain the following polynomials that are the generic solution of Eq. (1.3) for different α: $P^{\alpha=1}_{k=\overline{0,1}} = c_0 + c_1\eta$, $P^{\alpha=2}_{k=\overline{0,2}} = c_0(1-\eta^2) + c_1\eta$, $P^{\alpha=3}_{k=\overline{0,3}} = c_0(1-3\eta^2) + c_1(\eta - \eta^3/3)$, ... ($\alpha = \overline{1,N}$ and $k = \overline{0,\alpha}$). We write the solution of Eq. (1.1) in the form

$$U(x,y) = \sum_{\alpha=0}^{N} A_\alpha x^\alpha P^\alpha(\eta) = \sum_{\alpha=0}^{N} A_\alpha U_\alpha(x,y)$$

$$= A_0 c_{00} + A_1 x(c_{01} + c_{11}\eta) + A_2 x^2[c_{02}(1-\eta^2) + c_{12}\eta] + \ldots + A_N x^N P_k^N(\eta),$$

where A_α are arbitrary coefficients to be determined; the number of these coefficients depends on the method for solving the boundary-value problem and on the estimate of accuracy of the approximate solution.

The solutions found are generalized to n independent variables. The Laplace equation

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} + \ldots + \frac{\partial^2 U}{\partial x_n^2} = 0 \tag{1.5}$$

admits the invariant solutions

$$U = x_1^\alpha[J_2(\eta_2) + J_3(\eta_3) + \ldots + J_n(\eta_n)], \qquad \eta_2 = x_2/x_1, \ \ldots, \ \eta_n = x_n/x_1,$$

$$U = x_1^\alpha J(\eta), \qquad \eta = (x_2 + x_3 + \ldots + x_n)/x_1.$$

We write the generic polynomial solution of Eq. (1.5) for $n = 3$. We choose the invariant solution in the form [3]

$$U(x_1, x_2, x_3) = (a_1x_1 + b_1)^\alpha (a_2x_2 + a_3x_3 + b)^\beta J(\eta), \tag{1.6}$$

where $\eta = (a_2x_2 + a_3x_3 + b)/(a_1x_1 + b_1)$; a_1, a_2, a_3, and b_1 are arbitrary real or complex numbers (internal parameters).

Substituting (1.6) into (1.5), we obtain the reduced equation

$$\eta^2(\eta^2 + D^2)J'' - 2\eta[\eta^2(\alpha-1) - \beta D^2]J' + [\eta^2\alpha(\alpha-1) + \beta(\beta-1)D^2]J = 0$$

$[D^2 = (a_2^2 + a_3^2)/a_1^2]$. We write the solution of Eq. (1.5) for $n = 3$ in the form

$$U(x_1,x_2,x_3) = \sum_{\alpha=1-\beta}^{N-\beta} A_\alpha (a_1x_1 + b_1)^\alpha (a_2x_2 + a_3x_3 + b)^\beta P_k^\alpha(\eta, D) = \sum_{\alpha=1-\beta}^{N-\beta} A_\alpha U_\alpha(x_1,x_2,x_3,D),$$

where $P^{\alpha=1-\beta}_{k=\overline{0,1}} = \eta^{-\beta}(c_0 + c_1\eta)$, $P^{\alpha=2-\beta}_{k=\overline{0,2}} = \eta^{-\beta}[c_0(1-\eta^2/D^2) + c_1\eta]$, etc.

Similarly, we construct polynomial basis functions for the wave equation. Note, these polynomials coincide with polynomials for the Laplace equation if all minus signs are replaced by plus signs. The solutions found are generalized to n independent variables, i.e., to the equation

$$\frac{\partial^2 U}{\partial x_n^2} = \frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} + \ldots + \frac{\partial^2 U}{\partial x_{n-1}^2}. \tag{1.7}$$

The generic solution of Eq. (1.7) for $n = 4$ has the form

$$U(x_1,x_2,x_3,x_4) = (a_1x_1 + b_1)^\alpha (a_2x_2 + a_3x_3 + a_4x_4 + b)^\beta J(\eta), \tag{1.8}$$

where $\eta = (a_2x_2 + a_3x_3 + a_4x_4 + b)/(a_1x_1 + b_1)$. By direct verification, we can see that the solution

$$U(x_1,x_2,\ldots,x_n) = \sum_{\alpha=0}^{N} A_\alpha x_n^\alpha (c_{0\alpha}|\eta+1|^\alpha + c_{1\alpha}|\eta-1|^\alpha), \qquad \eta = \frac{x_1 + x_2 + \ldots + x_{n-1}}{x_n}$$

also satisfies Eq. (1.7). Here, α is an arbitrary real number; $c_{0\alpha}$ and $c_{1\alpha}$ are arbitrary constants.

Systems of basis functions for the Laplace equation and wave equation can be used to solve problems of statics and dynamics of the theory of elasticity with the use of generic solutions of the type of solutions found by Papkovich and Neuber, Galerkin, Trefftz, Sternberg and Eubanks, et al. [5, 6].

For the heat-conduction equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \tag{1.9}$$

we write the invariant solution in the form $U = t^{\alpha/2}J(\eta)$, where $\eta = x/\sqrt{t}$. Then, we obtain the solution of Eq. (1.9)

$$U(x,t) = \sum_{\alpha=0}^{N} A_\alpha t^{\alpha/2} P_k^\alpha(\eta),$$

where $P_{k=0}^{\alpha=0} = c_0$, $P_{k=1}^{\alpha=1} = c_1\eta$, $P_{k=0,2}^{\alpha=2} = c_0(1+\eta^2/2)$, $P_{k=1,3}^{\alpha=3} = c_1(\eta+\eta^3/6)$, etc.

The solutions found above are generalized to n independent variables. The invariant solution of the equation

$$\frac{\partial U}{\partial x_n} = \frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} + \ldots + \frac{\partial^2 U}{\partial x_{n-1}^2}$$

can be chosen, for instance, as

$$U = x_n^{\alpha/2}[J_1(\eta_1) + J_2(\eta_2) + J_3(\eta_3) + \ldots + J_{n-1}(\eta_{n-1})],$$

where $\eta_1 = x_1/\sqrt{x_n}$, $\eta_2 = x_2/\sqrt{x_n}$, $\eta_3 = x_3/\sqrt{x_n}$, …, $\eta_{n-1} = x_{n-1}/\sqrt{x_n}$.

We consider the algorithm for obtaining basis functions on the basis of homogeneous coordinates by the example of the solution of the Laplace equation. For Eq. (1.5), for $n = 3$, we write the solution in the form [3]

$$U(x_1, x_2, x_3) = \Phi(\xi, \eta) \tag{1.10}$$

($\xi = x_2/x_1$ and $\eta = x_3/x_1$). Substituting (1.10) into (1.5) for $x_1 \neq 0$, $\xi = i\xi^*$, and $\eta = i\eta^*$ ($i^2 = -1$), we obtain an equation of the form

$$(1-\xi^{*2})\frac{\partial^2\Phi}{\partial\xi^{*2}} - 2\xi^*\eta^*\frac{\partial\xi^{*2}}{\partial\xi^*\,\partial\eta^*} + (1-\eta^{*2})\frac{\partial\xi^*\,\partial\eta^*}{\partial\eta^{*2}} - 2\xi^*\frac{\partial\Phi}{\partial\xi^*} - 2\eta^*\frac{\partial\Phi}{\partial\eta^*} = 0.$$

This equation can be reduced to the wave equation and Laplace equation in new coordinates μ and ν by the following substitution:

— for $\mu = \eta^*/(\xi^*-1)$ and $\nu = \sqrt{\xi^{*2}+\eta^{*2}-1}/(\xi^*-1)$, we obtain the wave equation

$$\frac{\partial^2\Phi}{\partial\mu^2} - \frac{\partial^2\Phi}{\partial\nu^2} = 0;$$

— for $\mu = \eta^*/(\xi^*-1)$ and $\nu = \sqrt{1-\xi^{*2}-\eta^{*2}}/(\xi^*-1)$, we obtain the Laplace equation

$$\frac{\partial^2\Phi}{\partial\mu^2} + \frac{\partial^2\Phi}{\partial\nu^2} = 0.$$

Let us consider, e.g., the second case. Using the results obtained previously, we write the extended solution of Eq. (1.5)

$$U(x_1, x_2, x_3) = \sum_{\alpha=1}^{N} A_\alpha \mu^\alpha P^\alpha(\Theta),$$

where $\Theta = \nu/\mu$, $P_{\overline{1,k=0,1}}^{\alpha=1} = c_0 + c_1\Theta = c_0 + c_1 i\sqrt{x_1^2+x_2^2+x_3^2}/x_3$, $P_{\overline{1,k=0,2}}^{\alpha=2} = c_0(1-\Theta^2) + c_1\Theta = c_0(x_1^2 + x_2^2 + 2x_3^2)/x_3^2 + c_1 i\sqrt{x_1^2+x_2^2+x_3^2}/x_3$, etc.

The invariant solutions presented above are generalized to equations with n independent variables, if the solution is chosen in the form

$$U(x_1, x_2, \ldots, x_n) = \Phi_1\Big(\frac{x_2}{x_1}, \frac{x_3}{x_1}\Big) + \Phi_2\Big(\frac{x_2}{x_1}, \frac{x_4}{x_1}\Big) + \ldots + \Phi_{n-2}\Big(\frac{x_2}{x_1}, \frac{x_n}{x_1}\Big).$$

Note, the Laplace equation $\Delta U(x_1, x_2, x_3) = 0$ in choosing new independent variables $\xi = x_2/x_1$ and $\eta = x_3/x_1$ and conversion of the resultant equation to the canonical form transforms again to the Laplace equation [6] $\Delta U(\mu,\nu) = 0$, where μ and ν are expressed in terms of the homogeneous constants x_2/x_1 and x_3/x_1. By analogy with the two-dimensional space, we can introduce the function of a complex variable that depends from three rather than two independent variables: $W(\rho) = U(\mu,\nu) + iV(\mu,\nu)$, where $\rho(x_1, x_2, x_3) = \mu + i\nu$. In this case, the function $U(\mu,\nu)$ satisfies the Laplace equation, and the function $V(\mu,\nu)$ is found from the Cauchy–Riemann condition.

We show the relation of the previously found basis functions with hypergeometric Gaussian functions. For the wave equation (1.7) with $n = 4$, we choose an invariant solution of the form (1.8) for $\beta = 0$. Substituting (1.8) into (1.7), we obtain the reduced equation

$$(\eta^2 - D)J'' - 2\eta(\alpha - 1)J' + \alpha(\alpha - 1)J = 0, \tag{1.11}$$

where $D = (a_4^2 - a_2^2 - a_3^2)/a_1^2$.

Using the substitution $J = y(\xi)$, $\eta = -D + 2D\xi$, we transform Eq. (1.11) to the differential equation (in the normal Gaussian form)

$$\xi(1-\xi)y'' + [-(\alpha - 1) + 2(\alpha - 1)\xi]y' - \alpha(\alpha - 1)y = 0,$$

whose hypergeometric series has the form

$$F(-\alpha, -\alpha + 1, -\alpha + 1, \xi) = 1 - \alpha\xi - \frac{\alpha(-\alpha + 1)}{1 \cdot 2}\,\xi^2$$

$$- \ldots - \frac{\alpha(-\alpha + 1)(-\alpha + 2)\cdots(-\alpha + k - 1)}{k!}\,\xi^k - \ldots = \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{(1)^k}\,\xi^k.$$

If α is positive integer, the solution of Eq. (1.7) takes the form

$$U(x_1, x_2, x_3, x_4) = \sum_{\alpha=0}^{N} (a_4 x_4 + b_4) A_\alpha F(-\alpha, -\alpha + 1, -\alpha + 1, \xi), \tag{1.12}$$

where $\xi = \eta/(2D) - 1/2$. The solution of the type (1.12) is also valid for the Laplace equation (1.5) for $n = 3$, $\beta = 0$, and $D = (a_2^2 + a_3^2)/a_1^2$.

The expression in brackets $(\psi, g) = \sum_{k=1}^{n} \Big(\frac{\partial\psi}{\partial y_k}\,\frac{\partial g}{\partial x_k} - \frac{\partial\psi}{\partial x_k}\,\frac{\partial g}{\partial y_k}\Big)$ is called Poisson's bracket [6]. We equate this expression to zero, preliminary replacing g by $g = \partial\psi/\partial y_k$:

$$\sum_{k=1}^{n} \Big(\frac{\partial\psi}{\partial y_k}\,\frac{\partial^2\psi}{\partial x_k\,\partial y_k} - \frac{\partial\psi}{\partial x_k}\,\frac{\partial^2\psi}{\partial y_k^2}\Big) = 0. \tag{1.13}$$

We choose the invariant solution in the form

$$\psi = \sum_{k=1}^{n} (a_{1k} x_k + b_{1k})^\alpha J_k(\eta_k), \qquad \eta_k = \frac{a_{2k} y_k + b_{2k}}{a_{k1} x_k + b_{1k}}. \tag{1.14}$$

Substituting (1.14) into (1.13), we obtain

$$\sum_{k=1}^{n} \Big[(\alpha - 1)(J_k')^2 - \alpha J_k J_k''\Big] = 0.$$

The sought functions $J_k(\eta_k)$ in this expression are determined from identical differential equations whose solution is written in the form $J_k(\eta_k) = (\tilde{c}_{0k}\eta_k + \tilde{c}_{1k})^\alpha$ ($\tilde{c}_{0k}$ and $\tilde{c}_{1k}$ are integration constants). With allowance for the last expression, the invariant solution (1.14) takes the form

$$\psi = \sum_{k=1}^{n} (c_{1k} x_k + c_{0k} y_k + d_k)^\alpha,$$

where $d_k = c_{1k} b_{1k} + c_{0k} b_{2k}$, $c_{0k} = \tilde{c}_{0k} a_{2k}$, $c_{1k} = \tilde{c}_{1k} a_{1k}$, and α, c_{0k}, c_{1k}, and d_k are arbitrary real or complex numbers.

Owing to specific properties of Eq. (1.14), its solution for $k = 1$ ($x_1 = x$, $y_1 = y$) can be written in the form

$$\psi(x, y) = \sum_{\alpha=0}^{N} A_\alpha (c_1 x + c_0 y + d_1)^\alpha + \sum_{\alpha=1}^{N} B_\alpha \frac{1}{(c_1 x + c_0 y + d_2)^\alpha}, \tag{1.15}$$

where A_α and B_α are coefficients to be determined. To obtain the Laurent series from expression (1.15), we set $c_1 = 1$, $c_0 = i$, $d_1 = d_2 = -a$, $z = x + iy$, $N \to \infty$, $i^2 = -1$, and

$$A_\alpha \equiv a_k = \frac{1}{2\pi i} \oint_c \frac{1}{(\zeta - a)^{k+1}}\, f(\zeta)\, d\zeta, \qquad B_\alpha \equiv b_k = \frac{1}{2\pi i} \oint_c (\zeta - a)^{k-1} f(\zeta)\, d\zeta.$$

Then, we obtain the Laurent series

$$\psi(z) = \sum_{k=0}^{\infty} a_k (z-a)^k + \sum_{k=1}^{\infty} a_k (z-a)^{-k}.$$

We can easily show that the solution (1.15) satisfies the wave equation for $c_0 = c_1$ and the Laplace equation and biharmonic equation for $c_1 = ic_0$.

2. Solution of Inhomogeneous Equations with Variable Coefficients. We consider the equation

$$\phi_1(x,y)\,\frac{\partial^2 U}{\partial x^2} + \phi_2(x,y)\,\frac{\partial^2 U}{\partial y^2} = f(x,y), \tag{2.1}$$

where $\phi_1(x,y)$ and $\phi_2(x,y)$ are arbitrary functions that can be approximated by homogeneous polynomials $P_{\alpha-2}(\eta)$ (see Sec. 1):

$$\phi_1(x,y) = \sum_{\alpha=2}^{N} C_\alpha x^{\alpha-2} P_{\alpha-2}(\eta), \qquad \phi_2(x,y) = \sum_{\alpha=2}^{N} D_\alpha x^{\alpha-2} P_{\alpha-2}(\eta).$$

The function $f(x,y)$ is set by the expression

$$f(x,y) = B_0 + B_1 x P_1(\eta) + B_2 x^2 P_2(\eta) + B_3 x^3 P_3(\eta) + \ldots\ . \tag{2.2}$$

Here C_α, D_α, and B_α are specified approximation coefficients and $\eta = y/x$.

We seek the solution of Eq. (2.1) in the form

$$U = x^2 J_2(\eta) + x^3 J_3(\eta) + x^4 J_4(\eta) + \ldots\ . \tag{2.3}$$

We substitute (2.2) and (2.3) into (2.1) and equate the expressions at identical powers of x in the resultant equality. We obtain a system of inhomogeneous ordinary differential equations with respect to the sought functions J_α, whose solution is found sequentially beginning from $\alpha = 2$ (the solution of the homogeneous Laplace equation for $\alpha \geqslant 2$ is given in Sec. 1). Knowing the solutions of the homogeneous equation, we find the solution of the inhomogeneous differential equation. The method for solving equations of the type (2.1) is applied to canonical equations of mathematical physics with variable coefficients and also to all linear partial differential equations that admit a group of extensions (compressions) in terms of dependent and independent variables and a group of translations in terms of independent variables and is generalized to n independent variables.

3. Reduction of Nonlinear Boundary-Value Problems of Mechanics to Systems of Linear Algebraic Equations. For many equations and their systems with an appropriate choice of basis functions, the boundary-value problems considered are reduced to systems of linear algebraic equations (SLAE) without using traditional methods of linearization. Linearization is performed by choosing basis functions that are solutions of equations of mathematical physics and also the equation obtained on the basis of Poisson's bracket. Such equations and their systems include the Navier–Stokes equations for the potential flow in steady and unsteady cases, the Helmholtz equation, the equation and boundary conditions of the Plateau problem, the Monge–Ampère equations, and the Kármán system of equations [6].

We demonstrate this approach by constructing the algorithm for solving the Helmholtz equation

$$\frac{\partial F}{\partial \eta}\,\frac{\partial \Delta F}{\partial \xi} - \frac{\partial F}{\partial \xi}\,\frac{\partial \Delta F}{\partial \eta} = \Delta^2 F, \tag{3.1}$$

where Δ^2 is a biharmonic operator, $\psi(x,y) = \nu F(\xi,\eta)$, $\xi = xu_\infty/\nu$, and $\eta = yu_\infty/\nu$.

We supplement Eq. (3.1) by the following boundary conditions:

$$\frac{\partial F}{\partial \eta}\Big|_C = f_0(s), \qquad \frac{\partial F}{\partial \xi}\Big|_C = f_1(s), \qquad \frac{\partial F}{\partial \eta}\Big|_{\eta\to\infty} = 1, \qquad \frac{\partial F}{\partial \xi}\Big|_{\eta\to\infty} = 0$$

if the equation of the contour C is given in a parametric form $\xi = \xi(s)$ and $\eta = \eta(s)$.

We seek the solution of Eq. (3.1) in the form (see Sec. 1)

$$F(\xi,\eta) = \sum_{\alpha} A_\alpha (c_0\eta + c_1\xi + d)^\alpha + \eta \qquad (\alpha < 0),$$

where d, c_0, and c_1 are free internal parameters to be prescribed. Substituting this solution into the initial Eq. (3.1), in the domain G we obtain the linearized equation with respect to unknown coefficients A_α:

$$c_1(c_1^2+c_0^2)\sum_\alpha A_\alpha\alpha(\alpha-1)(\alpha-2)(c_0\eta+c_1\xi+d)^{\alpha-3}$$

$$=(c_1^2+c_0^2)^2\sum_\alpha A_\alpha\alpha(\alpha-1)(\alpha-2)(\alpha-3)(c_0\eta+c_1\xi+d)^{\alpha-4}.$$

We solve the problem by the method of weighted residues [6]. The longitudinal and transverse components of velocity of the liquid (gas) flow and the vorticity along the z axis are

$$v=\Big[-\sum_\alpha A_\alpha\alpha(c_0\eta+c_1\xi+d)^{\alpha-1}\Big]c_1u_\infty,\qquad u=\Big[c_0\sum_\alpha A_\alpha\alpha(c_0\eta+c_1\xi+d)^{\alpha-1}+1\Big]u_\infty,$$

$$\omega_z=\frac{1}{2}\Big(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\Big)\neq 0.$$

Note, if we replace c_0 by ic_1 ($i^2=-1$) in the solution $F(\xi,\eta)$, we obtain rational basis functions determined by separation of the imaginary and real parts. These rational basis functions satisfy Eq. (3.1) identically. In solving the problem by the method of weighted residues, the residues are formed only on the boundary.

Let us give another example. By introducing the stress function $\varphi(x,y)$ and the stream function $\psi(x,y)$, we can write the nonlinear differential equations that describe deformation of a rigidly plastic inhomogeneous body (with allowance for plasticity of the general type) in the form [7–10]

$$(\varphi_{yy}-\varphi_{xx})^2+4\varphi_{xy}^2=4k^2(x,y); \tag{3.2}$$

$$(\varphi_{yy}-\varphi_{xx})(\psi_{xx}-\psi_{yy})+(-4\varphi_{xy})\psi_{xy}=0, \tag{3.3}$$

where $k(x,y)$ is a known function (yield strength), $\sigma_x=\partial^2\varphi/\partial y^2$, $\sigma_y=\partial^2\varphi/\partial x^2$, $\tau_{xy}=-\partial^2\varphi/\partial x\,\partial y$, $u=\partial\psi/\partial y$, and $v=-\partial\psi/\partial x$. System (3.2), (3.3) should be supplemented by boundary conditions in stresses and displacements [9].

We introduce new independent variables

$$\xi=a_1x+\lambda_1y+d_1,\qquad \eta=b_2x+\lambda_2y+d_2, \tag{3.4}$$

where a_1, λ_1, d_1, b_2, λ_2, and d_2 are arbitrary real or complex numbers. Then, Eq. (3.2) takes the form

$$\Big[(\lambda_1^2+a_1^2)\varphi_{\xi\xi}+2\sqrt{(\lambda_1^2+a_1^2)(\lambda_2^2+b_2^2)}\,\varphi_{\xi\eta}-(\lambda_2^2+b_2^2)\varphi_{\eta\eta}\Big]^2$$

$$+4\varphi_{\xi\eta}\Big\{\Big[a_1b_2+\lambda_1\lambda_2-\sqrt{(\lambda_1^2+a_1^2)(\lambda_2^2+b_2^2)}\Big](\lambda_1^2+a_1^2)\varphi_{\xi\xi}$$

$$+\Big[a_1b_2+\lambda_1\lambda_2+\sqrt{(\lambda_1^2+a_1^2)(\lambda_2^2+b_2^2)}\Big](\lambda_2^2+b_2^2)\varphi_{\eta\eta}\Big\}+2(a_1b_2+\lambda_1\lambda_2)^2\varphi_{\xi\xi}\varphi_{\eta\eta}=4\bar{k}^2(\xi,\eta). \tag{3.5}$$

This approach allows obtaining equations of all three types, depending on the choice of parameters in transformation (3.4) and an appropriate combination of terms in expression (3.5). Let $a_1b_2+\lambda_1\lambda_2=0$; then, expression (3.5) reduces to the form

$$(\lambda_1^2+a_1^2)^2\Big[\varphi_{\xi\xi}-\Big(\frac{\lambda_2}{a_1}\Big)^2\varphi_{\eta\eta}+2\Big|\frac{\lambda_2}{a_1}\Big|\varphi_{\xi\eta}\Big]^2-4(\lambda_1^2+a_1^2)^2\Big|\frac{\lambda_2}{a_1}\Big|\varphi_{\xi\eta}\Big[\varphi_{\xi\xi}-\Big(\frac{\lambda_2}{a_1}\Big)^2\varphi_{\eta\eta}\Big]=4\bar{k}^2(\xi,\eta). \tag{3.6}$$

For simplicity, we assume that $a_1=\lambda_2$. Substituting the hyperbolic basis functions (see Sec. 1) into expression (3.6), we linearize the latter with respect to the sought function after extracting the square root.

In the general case, where some coefficients in transformation (3.4) are complex numbers, if the equality $a_1b_2+\lambda_1\lambda_2=\sqrt{(\lambda_1^2+a_1^2)(\lambda_2^2+b_2^2)}$ is satisfied, Eq. (3.5) can be written in the following form (e.g., $\lambda_1=0$, $\lambda_2=\sqrt{3/2}$, $b_2=2/\sqrt{2}$, and $a_1=\sqrt{3/2}+i\sqrt{2}/2$):

$$\Big[(\lambda_1^2+a_1^2)\varphi_{\xi\xi}+2\sqrt{(\lambda_1^2+a_1^2)(\lambda_2^2+b_2^2)}\,\varphi_{\xi\eta}-(\lambda_2^2+b_2^2)\varphi_{\eta\eta}\Big]^2+2(a_1b_2+\lambda_1\lambda_2)\varphi_{\eta\eta}(\gamma_1\varphi_{\xi\eta}+\gamma_2\varphi_{\xi\xi})=4\bar{k}^2(\xi,\eta). \tag{3.7}$$

Here $\gamma_1=4(\lambda_2^2+b_2^2)$ and $\gamma_2=a_1b_2+\lambda_1\lambda_2$.

We consider the equation

$$\gamma_1\varphi_{\xi\eta}+\gamma_2\varphi_{\xi\xi}=0. \tag{3.8}$$

Substituting the invariant solution $\varphi = \eta^\alpha J(\vartheta)$, $\vartheta = \xi/\eta$ into (3.8), we obtain the equation

$$\eta^{\alpha-2}[(\gamma_1 - \vartheta\gamma_2)J'' + \gamma_2(\alpha - 1)J'] = 0 \qquad (\eta^{\alpha-2} \neq 0),$$

whose solution is the function

$$J(\vartheta) = (-c_1/\gamma_2)(\gamma_1 - \gamma_2\vartheta)^\alpha + c_2,$$

where c_1 and c_2 are constants of integration and α is an arbitrary real or complex number. Then, the solution of Eq. (3.8) is written as

$$\varphi(\xi, \eta) = \sum_\alpha A_\alpha \Big[-\frac{c_1}{\gamma_2}\,(\gamma_1\eta - \gamma_2\xi)^\alpha + c_2\eta^\alpha \Big] \tag{3.9}$$

(A_α are coefficients to be determined). After substitution of (3.9) into (3.7), the latter is linearized with respect to the sought function. Having solved the problem in stresses, we solve Eq. (3.3) using, for instance, hyperbolic basis functions.

Finally, we note the following features of the basis functions presented in this work. The basis functions have a good structure and convenient analytical and computational properties. For instance, for the Laplace equation, the spherical harmonics have a scatter of coefficients within the range from 35/128 to 4,341,887,550, whereas the functions in the present work have a scatter from 1 to 126. In many cases, the spatial dimension is reduced by unity. The solutions are presented in an analytical form; therefore, the formulation and solution of the problems of parametric identification and inverse problems are simplified. Both linear and nonlinear mathematical models are reduced to linear algebraic equations. Knowing the analytical solution and boundary conditions, due to internal parameters (c_0, c_1, a_1, b_2, $\ldots$), one can eliminate all singularities associated with solving SLAE. After substitution of the basis functions (1.15) satisfying Eq. (1.13) into nonlinear differential equations belonging to a certain class, the latter are reduced to SLAE without using traditional methods of linearization.

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